The Primordial Curvature Perturbation from Vector Fields of General non-Abelian Groups

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We consider the generation of primordial curvature perturbation by general non-Abelian vector fields without committing to a particular group. Self-interactions of non-Abelian fields make the field perturbation non-Gaussian. We calculate the bispectrum of the field perturbation using the in-in formalism at tree level. The bispectrum is dominated by the classical evolution of fields outside the horizon. In view of this we show that the dominant contribution can be obtained from the homogeneous classical equation of motion. Then we calculate the power spectrum of the curvature perturbation. The anisotropy in spectrum is suppressed by the number of fields. This makes it possible for vector fields to be responsible for the total curvature perturbation in the Universe without violating observational bounds on statistical anisotropy. The bispectrum of the curvature perturbation is also anisotropic. Finally we give an example of the end-of-inflation scenario in which the curvature perturbation is generated by vector gauge fields through varying gauge coupling constant(s), which in covariant derivatives couples the Higgs field to the vector fields. We find that reasonably large gauge groups may result in the observable anisotropy in the power spectrum of the curvature perturbation.

I. INTRODUCTION

Inflation was proposed to alleviate the horizon and flatness problems of the Hot Big Bang cosmology [1, 2]. Currently it still is arguably the most compelling mechanism to explain the high degree of homogeneity and isotropy of the Universe as well as its flatness. The case for inflation was further strengthened after the release of first detailed observations of the Cosmic Microwave Background (CMB) radiation. The spectrum of the temperature perturbation of the CMB was found to be consistent with inflationary predictions and ruled out the rival theory of cosmic strings as the primary origin of this perturbation. Moreover, with the increasing precision of observational data it is becoming possible not only to falsify the competing theories of the origin of the primordial perturbation, but to falsify different models of inflation themselves. Measurements of temperature irregularities in the CMB sky provide a powerful tool to probe the physics of the very early Universe and the increasing precision allows to do it in a more and more detail.

The simplest inflationary models, with a scalar field driving inflation and producing the primordial curvature perturbation ζ, are still consistent with current CMB data. But several tentative anomalies, which are persistently found in all WMAP data releases, might suggest a need of more complex models. For example, the observed power asymmetry [3, 4], the alignment of low-l CMB multipoles [5–8] or a deep cold spot in the southern Galactic hemisphere [9, 10]. If the origin of these anomalies is confirmed to be primordial it will imply some degree of the statistical inhomogeneity and/or anisotropy of the primordial curvature perturbation. Such anomalies cannot be explained by the simplest inflationary models invoking only scalar fields. Scalar fields do not generate statistical anisotropy as they do not choose a preferred direction. But vector fields do, and if statistical anisotropy is established to be of primordial origin, the most natural way to explain it is with effects of vector fields.

There are two ways through which non-negligible contribution of vector fields to the evolution of the universe can induce statistical anisotropy. First, if a vector field have an effect on the global expansion of the universe, the latter will be anisotropic. The anisotropic expansion during inflation causes statistically anisotropic quantum fluctuations of light scalar fields. When these fluctuations become classical and cause perturbations in the metric, the latter are statistically anisotropic too as well as the temperature irregularities of CMB. This mechanism was first considered in [11]. On the other hand, the energy density of vector fields can be negligible, so that the global expansion is approximately isotropic, but perturbations of vector fields themselves generate or contribute to ζ. In general quantum fluctuations of vector fields are statistically anisotropic making their contribution to ζ statistically anisotropic too. The generation of ζ by vector fields was first considered in Ref. [12]. However, this and several subsequent papers [13, 14] did not consider statistical anisotropy, which was first considered in Ref. [15]. The comprehensive study of

statistical anisotropy generated by a vector field can be found in Ref. [16] (see also Ref. [17] for computations of anisotropic non-linearity parameter f_{NL}).

Recently the interest in statistical anisotropy and vector fields has grown considerably. Authors of Refs. [18–23] studied two-, three- and four-point correlators of the curvature perturbation generated by vector fields in detail. While in Refs. [24, 25] a model with a massive vector curvaton field is presented in which the vector field can generate both statistically isotropic and anisotropic perturbation. In these works a negligible contribution of vector fields to the global expansion rate is assumed.

Another line of research was concentrated on effects of anisotropic inflation on the curvature perturbation [26–28]. The universe during inflation expands anisotropically if the backreaction of the vector field is non-negligible. Such setup is considered in Refs. [11, 29–34]. Particularly interesting are the results of Refs. [32, 33]. Authors of these papers considered a model with time varying kinetic function of an Abelian vector field of the form $f(t) F_{\mu\nu} F^{\mu\nu}$, where $F_{\mu\nu}$ is the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It was shown that if $f(t)$ is modulated by the inflaton, then the scaling of the form $f \propto a^{-4}$ (a being a scale factor) is an attractor solution for a large parameter space. This is very significant as such scaling leads to the flat perturbation spectrum for the vector field [24, 25]. fF^2 coupling also induces anisotropy in inflationary expansion of the order of the slow-roll parameter. In addition in Ref. [33] it was shown that the vector field backreaction slows down the inflaton, i.e. the inflaton potential is effectively "flattened". The backreaction of non-Abelian vector fields with the time-varying kinetic function was also considered in Ref. [35].

The interest in detecting statistical anisotropy in the CMB is increasing too [36–41]. The anisotropy in the power spectrum of the primordial curvature perturbation can be parametrized as

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta^{\mathrm{iso}}(k) \left[1 + g_\zeta(k) \left(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} \right)^2 \right], \tag{1}$$

where only a quadrupole term is kept, $\hat{\mathbf{n}}$ and $\hat{\mathbf{k}}$ are unit vectors and the amplitude $g_\zeta(k)$ is in general a function of wavenumber k. First results of measuring g_ζ were given in Ref. [37]. After correcting a mistake in this work Refs. [38, 39] published consistent results detecting the departure from statistical isotropy at very high 9σ significance level $g_\zeta = 0.29 \pm 0.031$. They also found that the preferred direction $\hat{\mathbf{n}}$ is very close to the ecliptic pole $(l, b) = (96, 30)$. The proximity of $\hat{\mathbf{n}}$ to the ecliptic pole suggests very strongly that the detected g_ζ is due to a systematic effect. This was indeed discussed in Refs. [39, 40] were they investigated whether a non-zero g_ζ could be accounted for by the WMAP beam asymmetry or other systematics. Unfortunately, both works reached somewhat contradicting conclusions. The authors of Ref. [37] could not determine a systematic effect causing such a large g_ζ, while Ref. [40] claims it is due to the beam asymmetry and give the bound $|g_\zeta| < 0.07$. Despite this, as the precision of measurements increase, the prospect of detecting statistical anisotropy of primordial origin is very exciting. Such anisotropy offers a new observable to probe the physics of the very early Universe. Indeed, as shown in Refs. [36, 41] with the Planck temperature data alone it will be possible to constrain g_ζ with an accuracy of 0.01 (2σ). In addition, the polarization data alone will offer 0.03 accuracy as the consistency check. Furthermore, an extended Planck mission can constrain the spectral index of $g_\zeta \propto k^q$ to an accuracy of $\Delta q \sim 0.3$ (1σ).

The interest in measuring the anisotropy of higher order correlators is increasing too [42–44]. This is justified, because higher order correlators can be predominantly anisotropic as shown in Refs. [15–19, 21, 22, 24, 25]. This was specifically emphasized in Refs. [17, 24, 25, 45] were non-linearity parameter f_{NL} was calculated for several models. It was found that even if anisotropy in the power spectrum is subdominant, f_{NL} can be predominantly anisotropic with the same preferred direction $\hat{\mathbf{n}}$ as the spectrum. Moreover, the magnitude of f_{NL} is proportional to anisotropy in the spectrum g_ζ. If such non-Gaussianity is detected it would be a smoking gun for the vector field contribution to the primordial curvature perturbation. Impacts of statistically anisotropic primordial perturbation on CMB observables were studied in Refs. [28, 46].

Instead of vector fields being responsible only for the curvature perturbation, Refs. [47–52] also propose models in which vector fields drive inflation. Such models face a challenge of making expansion of the universe predominantly isotropic. The two suggestions proposed are either introducing an orthogonal triad of vector fields or a large number of randomly oriented ones. Particularly, authors of Refs. [51, 52] also consider non-Abelian vector fields. Their setup consists of an orthogonal triad of equal norm vector fields of $SU(2)$ group. In this case the total energy-momentum tensor of vector fields is isotropic and if they dominate the universe, the latter inflates isotropically.

The study of vector fields in inflationary cosmology are not only interesting from the phenomenological point of view, as means of explaining CMB anomalies. Such models are also interesting from the theoretical perspective. The possibility of vector fields affecting or generating the total curvature perturbation in the Universe opens a new window for inflationary model building. We might no longer need direct involvement of scalars to create ζ; it could be created by vector fields with, for example, varying couplings. In particle physics models non-Abelian vector fields are much more common than Abelian ones. In addition, as large non-Abelian gauge groups have many vector fields, assuming their random orientation, it is natural to expect the suppression of statistical anisotropy of ζ.

In this paper we study the generation of ζ by non-Abelian vector fields. The effects of $SU(2)$ non-Abelian vector fields on the curvature perturbation were first studied in Refs. [21–23]. In this paper we demonstrate that non-Gaussian correlators are dominated by the interactions outside horizon. First, we make use of the full quantum formalism, the so called "in-in formalism", to calculate the bispectrum at tree level and show that it is dominated by the classical part. That is, by interactions of fields after horizon crossing. Since this is the case, correlation functions can be calculated much easier using only the homogeneous equation of motion of vector fields. We perform this calculation explicitly and show that the result obtained is the same as the dominant part of the result calculated in the in-in formalism.

We also calculate the spectrum and bispectrum of the curvature perturbation. It is found that the anisotropy in the spectrum of ζ is suppressed by the number of vector fields involved in generating ζ (assuming these are oriented randomly). Thus with a large enough gauge group ζ can be generated solely from vector fields without violating observational bound on the statistical anisotropy. However, it is possible that a small detectable anisotropy remains.

The bispectrum of ζ from non-Abelian vector fields is also anisotropic. Although the form of anisotropy is more complicated than in the single field case as it involves not one but many preferred directions.

In the last section we give a simple example of a non-Abelian gauge field generating ζ. We use the end-of-inflation scenario in which ζ is generated by the gauge fields through varying gauge coupling constant(s) in the covariant derivative, which couples Higgs field to the gauge fields.

The paper is organized as follows. In section II we define the Lagrangian and the setup of our model. In section III the bispectrum at tree level is calculated using the quantum in-in formalism. In section IV it is shown that the classical calculation of the bispectrum from the homogeneous equation of motion gives exactly the same result as the dominant part of the bispectrum in the previous section. Using the δN formula, the correlators of ζ are calculated in section V. Finally, an example of a mechanism for the generation of ζ from non-Abelian gauge fields is given in section VI. It is generated through the gauge couplings of the Higgs field. A summary is presented in section VII.

In this paper we use natural units, where $c = \hbar = 1$ and Newton's gravitational constant is $8\pi G = m_{\mathrm{Pl}}^{-2}$, where m_{Pl} is the reduced Planck mass.

II. THE LAGRANGIAN

Consider a general Lagrangian of non-Abelian vector fields

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}^a F_a^{\mu\nu}, \tag{2}$$

where the field strength tensor $F_{\mu\nu}^a$ is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \mathrm{f}^{abc} A_\mu^b A_\nu^c. \tag{3}$$

We do not specify the gauge group, f^{abc} are structure constants of the Lie algebra of any non-Abelian group and they are antisymmetric in permutations of indices a, b and c.

In this paper we are interested in a time varying g but we do not specify the origin of this variation. The modulation of g might be due to some scalar degree of freedom. The inflaton itself can modulate g (for such models with Abelian vector field see Refs. [29, 31–33] and non-Abelian ones see Ref. [35]), but in this paper we do not need to specify the origin of time dependence of g.

Let us recast the Lagrangian in Eq. (2) by factoring $g(t)$ into a constant and time dependent parts in the following way

$$g(t) \equiv g_{\mathrm{c}}/\sqrt{f(t)}. \tag{4}$$

g_{c} is the constant value of g when the modulating degree of freedom is stabilized at time t_{s}. At this moment the kinetic function $f(t_{\mathrm{s}}) = 1$. The constant g_{c} can be absorbed into the field strength tensor by the field redefinition $\tilde{A}_\mu^a \equiv A_\mu^a/g_{\mathrm{c}}$. Dropping tildes the Lagrangian in Eq. (2) then becomes

$$\mathcal{L} = -\frac{1}{4} f \, F_{\mu\nu}^a F_a^{\mu\nu}, \tag{5}$$

where the redefined field strength tensor becomes

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_{\mathrm{c}} \mathrm{f}^{abc} A_\mu^b A_\nu^c \tag{6}$$

with g_{c} being a self-coupling constant.

Using the gauge freedom of the non-Abelian vector field in Eq. (5) we can always choose a gauge in which any given space-time component of all the vector fields is zero [53]. However, in general it is not possible to make all four components of all vector fields in a group vanish by a gauge choice. This is in contrast to the Abelian field, where one can always choose a gauge in which the classical or homogeneous part of the field is zero. For the rest of the paper let us choose the vanishing component to be the temporal one so that $A_0^a = 0$ for all a. For the remainder of the paper we use the convention of space indices denoted by subscripts and gauge ones by superscripts except in those cases where equations are written in Lorentz covariant four-vector form.

The energy-momentum tensor of the massless vector field is anisotropic [12]. If the energy density of such a field is non-negligible the expansion of the universe becomes anisotropic. To avoid excessive large scale anisotropy one can introduce a large number of vector fields, suppressing the anisotropy by a factor of $\sqrt{\mathcal{N}}$, where $\mathcal{N}$ is the number of vector fields. Such mechanism is employed in vector inflation [48]. In our setup this could be achieved by taking a very large gauge group. Another possibility is to introduce three identical, orthogonal vector fields [48, 54]. This option was recently explored with $SU(2)$ vector fields in Refs. [51, 52]. In this paper we consider a third possibility, namely we assume that the energy density of the vector field is negligible during inflation. In other words, we neglect a backreaction of the vector field on the expansion of the universe. However vector fields can still generate the curvature perturbation. This might happen, for example, in the vector curvaton [12] or end-of-inflation [15] scenarios. In the former case, the non-Abelian vector fields must acquire a mass through a Higgs mechanism prior to generating ζ. An example of the latter case will be given in section VI.

Taking the contribution of the vector field to the total energy density to be negligible, inflation can be assumed to be isotropic and we can use the Friedmann-Lemaître-Robertson-Walker (FLRW) background with the metric $g_{\mu\nu} = \mathrm{diag}\left[1, -a^2(t), -a^2(t), -a^2(t)\right]$, where t is the cosmic time. We will also use the conformal time $\tau \equiv \int \mathrm{d}t/a(t)$ in the paper.

Expanding the Lagrangian in Eq. (5) in the FLRW background we find a term

$$\mathcal{L} \supset a^{-4} g_{\mathrm{c}}^2 \mathrm{f}^{abc} \mathrm{f}^{ade} A_i^b A_j^c A_i^d A_j^e, \tag{7}$$

in which the scale factor appears explicitly. But the normalization of the scale factor is arbitrary while the Lagrangian is a physical quantity related to the energy of the system and cannot contain arbitrary normalizable factors. The appearance of a in Eq. (7) is due to the fact that the vector field A_i^a is defined with respect to the comoving coordinates. The physical vector field, defined with respect to the physical coordinates is A_i^a/a [12, 16, 45]. Therefore, it will be useful to define a physical, canonically normalized vector field

$$W_i^a = \sqrt{f}\frac{A_i^a}{a}. \tag{8}$$

We also use Fourier modes $\delta W_i^a(\mathbf{k})$ of the perturbation of W_i^a

$$\delta W_i^a(\mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \delta W_i^a(\mathbf{k}) \, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}. \tag{9}$$

III. CORRELATION FUNCTIONS IN THE IN-IN FORMALISM

A. The in-in Formalism

In this section we calculate the three point correlation function of vector field perturbation. Following Refs. [55, 56] we use the so called "in-in formalism". In our setup, the calculation is simplified due to the assumption that the energy density of vector fields during inflation is negligible, which means that it is not necessary to consider the corresponding perturbation of the metric.

In the in-in formalism expectation values $\langle 0|\hat{O}|0\rangle$ are calculated in the interaction picture. In this picture both, field operators and state vectors, varies with time. The variation of the former is governed by the equation of motion of the free field, while the latter evolves due to interaction terms. To find free field operators for non-Abelian vector fields we note that each massless vector field has two degrees of freedom. The field is quantized by promoting these degrees of freedom to operators with canonical commutation relations. Thus $\delta W_i^a(\mathbf{k})$ in Eq. (9) becomes

$$\delta\hat{W}_i^a(\mathbf{k}, \tau) = \sum_{\lambda=\mathrm{L,R}} \left[e_i^\lambda\left(\hat{\mathbf{k}}\right) w(k, \tau) \hat{a}_\lambda^a(\mathbf{k}) - e_i^{\lambda*}\left(-\hat{\mathbf{k}}\right) w^*(k, \tau) \hat{a}_\lambda^{a\dagger}(-\mathbf{k}) \right], \tag{10}$$

where τ is the conformal time. The raising and lowering operators in this equation satisfy canonical commutation relations

$$\left[a_\lambda^a \left(\mathbf{k}\right), a_{\lambda'}^{b\dagger} \left(-\mathbf{k}'\right) \right] = (2\pi)^3 \delta \left(\mathbf{k} + \mathbf{k}'\right) \delta_{\lambda\lambda'} \delta_{ab} \tag{11}$$

with others being zero, while the left- and right-handed circular polarization vectors e_i^λ are chosen in such a way that $e_i^\lambda \left(-\hat{\mathbf{k}}\right) = -e_i^{\lambda*} \left(\hat{\mathbf{k}}\right)$. With $\hat{\mathbf{k}} = (0, 0, k)$ they become

$$e_i^{\mathrm{L}} \left(\hat{\mathbf{k}}\right) = \frac{1}{\sqrt{2}} (1, i, 0) \quad \text{and} \quad e_i^{\mathrm{R}} \left(\hat{\mathbf{k}}\right) = \frac{1}{\sqrt{2}} (1, -i, 0). \tag{12}$$

In the interaction picture $\delta\hat{W}_i^a \left(\mathbf{k}, \tau\right)$ is a free quantum field. Thus the conjugate pair $\{w, w^*\}$ in Eq. (10) are solutions of a free vector field equation of motion. In principle $\{w, w^*\}$ should have polarization and group indices, e.g. w_λ^a. However, the Lagrangian in Eq. (5) does not have parity violating terms and both polarization modes satisfy the same equation of motion. The same is true for the gauge index, all vector fields satisfy the same free field equation of motion with the same initial conditions. Thus to simplify the notation we dropped polarization and gauge indices out.

In Refs. [24, 25] it was found that w satisfies the same equation of motion as the scalar field if the kinetic function during inflation varies with time as $f \propto a^{-1\pm3}$. For a massless field in de Sitter background this equation becomes

$$\ddot{w} + 3H\dot{w} + \left(\frac{k}{a}\right)^2 w = 0, \tag{13}$$

where dots denote derivatives with respect to the cosmic time t. The solution of this equation is very well known. With Bunch-Davies vacuum initial conditions it is given in conformal time by

$$w = \frac{H}{\sqrt{2k^3}} \left(1 - ik\tau\right) \mathrm{e}^{-ik\tau}. \tag{14}$$

With these definitions the Wightman function becomes

$$\left\langle 0 \left| \delta\hat{W}_i^a \left(\mathbf{k}, \tau\right) \delta\hat{W}_j^b \left(\mathbf{k}', \tau'\right) \right| 0 \right\rangle = (2\pi)^3 \delta \left(\mathbf{k} + \mathbf{k}'\right) \delta_{ab} T_{ij}^{\mathrm{E}} \left(\hat{\mathbf{k}}\right) w \left(k, \tau\right) w^* \left(k, \tau'\right), \tag{15}$$

where the tensor $T_{ij}^{\mathrm{E}} \left(\hat{\mathbf{k}}\right)$ is defined by [16, 45]

$$T_{ij}^{\mathrm{E}} \left(\hat{\mathbf{k}}\right) \equiv e_i^L \left(\hat{\mathbf{k}}\right) e_j^R \left(\hat{\mathbf{k}}\right) + e_i^{\mathrm{R}} \left(\hat{\mathbf{k}}\right) e_j^{\mathrm{L}} \left(\hat{\mathbf{k}}\right) = \delta_{ij} - \hat{k}_i \hat{k}_j. \tag{16}$$

N-point correlation functions of the vector field perturbation are calculated in the interaction picture as vacuum expectation values of the form

$$g_N \left(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\right) = \left\langle 0 \left| \hat{U}^{-1} \delta\hat{W}_l^f \left(\mathbf{x}_1, \tau\right) \delta\hat{W}_m^g \left(\mathbf{x}_2, \tau\right) \ldots \delta\hat{W}_n^h \left(\mathbf{x}_N, \tau\right) \hat{U} \right| 0 \right\rangle, \tag{17}$$

where, to simplify notation, we also suppressed gauge and space indices in the function g_N. The unitary operator $\hat{U}$ is given by

$$\hat{U} = \exp \left\{ -i \int_{\tau_0}^\tau \hat{H}_{\mathrm{int}} \left(\tau'\right) \mathrm{d}\tau' \right\} \tag{18}$$

with τ_0 being some early time when the mode of interest is deep within the horizon. The interaction Hamiltonian $\hat{H}_{\mathrm{int}}$ can be found from the interaction terms of the Lagrangian in Eq. (5). With the temporal gauge ($A_0^a = 0$) these terms become

$$\mathcal{L}_{\mathrm{int}} = -a^{-4} \frac{1}{2} f \left[g_{\mathrm{c}} \mathrm{f}^{abc} \left(\partial_i A_j^a - \partial_j A_i^a\right) A_i^b A_j^c + \frac{1}{2} g_{\mathrm{c}}^2 \mathrm{f}^{abc} \mathrm{f}^{ade} A_i^b A_j^c A_i^d A_j^e \right]. \tag{19}$$

In this paper we calculate the three point correlation function at tree level, thus only the third order interaction Hamiltonian is considered. For the physical, canonically normalized field it is $\hat{H}_{\mathrm{int}} \equiv \hat{H}_{\mathrm{int}}^{(3)} + \hat{H}_{\mathrm{int}}^{(4)}$, where

$$\hat{H}_{\mathrm{int}}^{(3)} = a^3 \left(\tau\right) \int \mathrm{d}^3\mathbf{x} \, \frac{g_{\mathrm{c}}}{\sqrt{f}} \mathrm{f}^{abc} \partial_i \delta\hat{W}_j^a \delta\hat{W}_i^b \delta\hat{W}_j^c, \tag{20}$$

$$\hat{H}_{\mathrm{int}}^{(4)} = a^4 \left(\tau\right) \int \mathrm{d}^3\mathbf{x} \, \frac{1}{2} \frac{g_{\mathrm{c}}^2}{f} \left(\mathrm{f}^{abc} \mathrm{f}^{ade} + \mathrm{f}^{adc} \mathrm{f}^{abe}\right) W_i^b \delta\hat{W}_j^c \delta\hat{W}_i^d \delta\hat{W}_j^e. \tag{21}$$

The factor a^4 in these equations is due to $\sqrt{-\det\left[g_{\mu\nu}\right]}$.

As it is clear from Eqs. (20) and (21) $g\left(t\right) \equiv g_{\mathrm{c}}/\sqrt{f\left(t\right)}$ in Eq. (4) is the strength of self-coupling for the canonically normalized vector field. To keep quantum calculations under control this coupling must be ensured to be small. It must also be small for quantum fluctuations of interacting fields to become classical after horizon exit [57]. Furthermore, the perturbation spectrum of massless non-interacting vector fields is flat if the kinetic function is of the form $f \propto a^{-1\pm 3}$ [24, 25]. In order to preserve approximate flatness of the perturbation spectrum, interaction terms must be small. Because the variation of $f\left(t\right)$ is exponential at a time $t < t_{\mathrm{s}}$ with $f\left(t_{\mathrm{s}}\right) = 1$ by definition, $g = g_{\mathrm{c}}/\sqrt{f}$ is small only when $f \propto a^{-4}$. We will assume this to be the case for the rest of the paper.

With the weak self-coupling Eq. (18) can be expanded in powers of $g_{\mathrm{c}}/\sqrt{f}$. As we are interested in the tree level contribution to the bispectrum it is enough to keep only the first order term in Eq. (17) after which it becomes [56, 58]

$$g_3 = -i \int_{-\infty}^{0} \mathrm{d}\tau' \left\langle 0 \left| \left[\delta\hat{W}_l^f\left(\mathbf{x}_1,\tau\right) \delta\hat{W}_m^g\left(\mathbf{x}_2,\tau\right) \delta\hat{W}_n^h\left(\mathbf{x}_3,\tau\right), \hat{H}_{\mathrm{int}}\left(\tau'\right)\right]\right|0\right\rangle. \tag{22}$$

In what follows $g_3^{(3)}$ will denote the three point function with the Hamiltonian in Eq. (20) and $g_3^{(4)}$ in Eq. (21), so that the total function is $g_3 = g_3^{(3)} + g_3^{(4)}$.

B. The Three Point Correlation Function from the Quartic Term

Let us consider first the $g_3^{(4)}$ term. Going to the momentum space we find

$$g_3^{(4)}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right) = \left(\mathrm{f}^{abc}\mathrm{f}^{ade} + \mathrm{f}^{adc}\mathrm{f}^{abe}\right) W_i^b \frac{g_{\mathrm{c}}^2}{f_0} \int_{\tau_0}^{\tau_{\mathrm{end}}} \mathrm{d}\tau' a^8\left(\tau'\right) \int \frac{\mathrm{d}^3 q_1 \mathrm{d}^3 q_2 \mathrm{d}^3 q_3}{\left(2\pi\right)^6} \delta\left(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3\right) \times$$

$$\times \mathrm{Re}\left[-i\left\langle 0\left|\delta W_l^f\left(\mathbf{k}_1,\tau\right) \delta W_m^g\left(\mathbf{k}_2,\tau\right) \delta W_n^h\left(\mathbf{k}_3,\tau\right) \delta W_j^c\left(\mathbf{q}_1,\tau'\right) \delta W_i^d\left(\mathbf{q}_2,\tau'\right) \delta W_j^e\left(\mathbf{q}_3,\tau'\right)\right|0\right\rangle\right] \tag{23}$$

where $\mathrm{Re}\left[\ldots\right]$ denotes the real part and τ_{end} is the conformal time at the end of inflation. In this expression we used the fact that W_i^b is slowly rolling due to the smallness of self coupling term (more on this in section IV B). We have also used $f = f_0 a^{-4}$, where f_0 is some initial value. The correlator can be evaluated using Wick's theorem. With Eq. (15) after some tedious algebra we find

$$g_3^{(4)}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\right) = -\left(2\pi\right)^3 \delta\left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3\right) \frac{2H^6}{\prod_i^3 2k_i^3} \mathcal{T}_{lmn}^{(4)fgh}\left(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2, \hat{\mathbf{k}}_3\right) I^{(4)}\left(k_1, k_2, k_3\right). \tag{24}$$

The function $\mathcal{T}_{lmn}^{(4)fgh}$ depends only on the direction of three vectors $\left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\right)$ and thus quantifies the anisotropy of the three point correlator function. The full expression is given by

$$\mathcal{T}_{lmn}^{(4)fgh}\left(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2, \hat{\mathbf{k}}_3\right) \equiv W_m^b T_{lj}^{\mathrm{E}}\left(\hat{\mathbf{k}}_1\right) T_{nj}^{\mathrm{E}}\left(\hat{\mathbf{k}}_3\right) \left(\mathrm{f}^{abh}\mathrm{f}^{agf} + \mathrm{f}^{agh}\mathrm{f}^{abf}\right) +$$

$$+ W_l^b T_{mj}^{\mathrm{E}}\left(\hat{\mathbf{k}}_2\right) T_{nj}^{\mathrm{E}}\left(\hat{\mathbf{k}}_3\right) \left(\mathrm{f}^{abg}\mathrm{f}^{afh} + \mathrm{f}^{afg}\mathrm{f}^{abh}\right) +$$

$$+ W_n^b T_{lj}^{\mathrm{E}}\left(\hat{\mathbf{k}}_1\right) T_{mj}^{\mathrm{E}}\left(\hat{\mathbf{k}}_2\right) \left(\mathrm{f}^{abg}\mathrm{f}^{ahf} + \mathrm{f}^{ahg}\mathrm{f}^{abf}\right), \tag{25}$$

where $T_{ij}^{\mathrm{E}}\left(\hat{\mathbf{k}}\right)$ is defined in Eq. (16). $I^{(4)}$ is the integral of the form

$$I^{(4)} = \frac{g_{\mathrm{c}}^2}{f_0} \mathrm{Re}\left[i \int_{\tau_0}^{\tau_{\mathrm{end}}} \mathrm{d}\tau' a^8 \left(1 - ik_1\tau'\right)\left(1 - ik_2\tau'\right)\left(1 - ik_3\tau'\right) \mathrm{e}^{ik_{\mathrm{t}}\tau'}\right], \tag{26}$$

with $k_{\mathrm{t}} \equiv k_1 + k_2 + k_3$. This integral is calculated in Appendix A. Assuming all three k's crosses the horizon at a similar time it is equal to

$$I^{(4)} = \frac{g_{\mathrm{c}}^2}{f_0} \frac{k_{\mathrm{t}}^7 H^{-8}}{4!} \left[6\mathrm{e}^{4N_k}\left(\frac{1}{3} - K_1 + K_2\right) + 2\mathrm{e}^{2N_k}\left(K_1 - 3K_2 - \frac{1}{5}\right) - \right.$$

$$\left. - \left(\gamma + N_k\right)\left(\frac{1}{5}K_1 - K_2 - \frac{1}{35}\right) + \frac{1}{300}\left(625K_2 - 137K_1 + \frac{1019}{49}\right)\right], \tag{27}$$

where

$$K_1 \equiv \frac{\sum_{i>j}^3 k_i k_j}{k_t^2} \quad \text{and} \quad K_2 \equiv \frac{\prod_i^3 k_i}{k_t^3}, \tag{28}$$

$\gamma \approx 0.577$ is Euler-Mascheroni constant and $N_k \equiv -\ln(|k_t \tau_{\text{end}}|)$ is the number of e-folds from when k_t exits the horizon to the end of inflation. For the cosmological scales $N_k \sim 60$. We recognize the dominant term to be proportional to the dominant term of the three point correlation functions in Refs. [58, 59]. This term is the contribution to the correlation function from the superhorizon evolution of the fields. The set up in our case is somewhat different. The authors of Refs. [58, 59] considered a field with constant strength of self-coupling, while in our case the self-coupling of canonically normalized fields is varying with time, $g = g_c/\sqrt{f} \propto a^2$. From Eq. (27) we see that this variation enhances additional modes, both, when they are created at the horizon exit and during the evolution of the field outside the horizon. However, these modes are subdominant. Taking only the dominant contribution to $g_3^{(4)}$ we find

$$g_3^{(4)} = -(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \mathcal{T}_{lmn}^{(4)fgh}(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2, \hat{\mathbf{k}}_3) \frac{\sum_i^3 k_i^3}{\prod_i k_i^3} \frac{g_c^2}{f_{\text{end}}} \frac{H^2}{48}. \tag{29}$$

To evaluate this equation we used $f_{\text{end}} = f_k \exp(-4N_k)$, where f_k and f_{end} are the values of f at the horizon crossing and the end of inflation respectively.

C. The Three Point Correlation Function from the Cubic Term

From Eq. (20) we expect that the three point correlation function from the cubic term $g_3^{(3)}$ is suppressed by a factor of p compared to $g_3^{(4)}$, where $p = k/a$ is the modulus of the physical momentum. In this section we show that this is indeed the case. $g_3^{(3)}$ is calculated along the same lines as $g_3^{(4)}$. Taking a Fourier transform of Eq. (20) from Eq. (22) we find

$$\begin{aligned} g_3^{(3)} &= \frac{g_c}{\sqrt{f_0}} f^{abc} \int_{\tau_0}^{\tau_{\text{end}}} d\tau' \, a^5(\tau') \int \frac{d^3 q_1 d^3 q_2 d^3 q_3}{(2\pi)^6} \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) q_{1i} \times \\ &\quad \times 2i \, \text{Im}\left[\left\langle 0 \left| \delta W_l^f(\mathbf{k}_1, \tau) \delta W_m^g(\mathbf{k}_2, \tau) \delta W_n^h(\mathbf{k}_3, \tau) \delta W_j^a(\mathbf{q}_1, \tau') \delta W_i^b(\mathbf{q}_2, \tau') \delta W_j^c(\mathbf{q}_3, \tau') \right| 0 \right\rangle\right], \end{aligned} \tag{30}$$

where $\text{Im}[\ldots]$ denotes the imaginary part. Using Wick's theorem and Eq. (15) we calculate

$$g_3^{(3)} = -(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \frac{2H^6}{\prod_i^3 2k_i^3} \mathcal{T}_{lmn}^{(3)fgh}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) I^{(3)}(k_1, k_2, k_3), \tag{31}$$

where the anisotropy of the three point correlation function is given by

$$\begin{aligned} \mathcal{T}_{lmn}^{(3)fgh}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= f^{fgh}\left[T_{lj}^{\text{E}}(\hat{\mathbf{k}}_1) T_{mi}^{\text{E}}(\hat{\mathbf{k}}_2) T_{jn}^{\text{E}}(\hat{\mathbf{k}}_3)(\mathbf{p}_1 - \mathbf{p}_3)_i + \right. \\ &\quad + T_{lj}^{\text{E}}(\hat{\mathbf{k}}_1) T_{jm}^{\text{E}}(\hat{\mathbf{k}}_2) T_{ni}^{\text{E}}(\hat{\mathbf{k}}_3)(\mathbf{p}_2 - \mathbf{p}_1)_i + \\ &\quad \left. + T_{li}^{\text{E}}(\hat{\mathbf{k}}_1) T_{mj}^{\text{E}}(\hat{\mathbf{k}}_2) T_{jn}^{\text{E}}(\hat{\mathbf{k}}_3)(\mathbf{p}_3 - \mathbf{p}_2)_i \right]. \end{aligned} \tag{32}$$

$\mathbf{p} \equiv \mathbf{k}/a_{\text{end}}$ in this expression is the physical momentum, a_{end} is the scale factor at the end of inflation and

$$I^{(3)} \equiv -ia_{\text{end}} \frac{g_c}{\sqrt{f_0}} \text{Im}\left[\int_{\tau_0}^{\tau_{\text{end}}} d\tau' \, a^5(\tau')(1 - ik_1\tau')(1 - ik_2\tau')(1 - ik_3\tau') e^{ik_t\tau'}\right]. \tag{33}$$

Using the method explained in Appendix A it is calculated to be

$$I^{(3)} = -ia_{\text{end}} \frac{g_c}{\sqrt{f_0}} k_t^4 H^{-5}\left[e^{N_k}\left(\frac{1}{3} - K_1 + K_2\right) - \frac{\pi}{4}\left(\frac{1}{4} - K_1 + 2K_2\right)\right]. \tag{34}$$

The first, dominant term, is due to the evolution after horizon exit. Neglecting the subdominant term $g_3^{(3)}$ becomes

$$g_3^{(3)} = i(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \mathcal{T}_{lmn}^{(3)fgh}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \frac{\sum_i^3 k_i^3}{\prod_i k_i^3} \frac{g_c}{\sqrt{f_{\text{end}}}} \frac{H^2}{12}. \tag{35}$$

The anisotropic term $\mathcal{T}^{(4)}$ from quartic interactions in Eq. (25) is proportional to the homogeneous part of the vector field $W > H$,[1] while $\mathcal{T}^{(3)}$ from cubic interactions is proportional to the physical momentum p, which for cosmological scales are $p \ll H$. Thus $\left|g_3^{(4)}\right| \gg \left|g_3^{(3)}\right|$ and the dominant contribution to the three point correlation function is from quartic terms. Moreover, the dominant contribution in $g^{(4)}$ itself is from the classical evolution of fields.

IV. BISPECTRUM FROM THE CLASSICAL EVOLUTION

A. Classicality

In the last section we used perturbative quantum field theory to calculate correlators of the field perturbation. As the results in Eqs. (29) and (35) show those correlators are dominated by the interaction of fields after horizon exit. Furthermore, as correlators with derivative couplings are suppressed by the factor $k/a \ll H$, it suggests that we can obtain the dominant contribution to correlators by a simpler method: from the classical equation of motion of the homogeneous field.

It is well known that after a mode $\mathbf{k}$ of a free light quantum field crosses the horizon, i.e. when $k/aH \to 0$, the phase of the mode function becomes constant and field operator in the Heisenberg picture can be written in a form $\chi_k(\tau)\left(\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}}\right)$, where χ_k is made real by an arbitrary phase rotation [60, 61]. In this limit all commutators of fields vanish and the eigenvector of a field operator at some particular time remains an eigenvector thereafter. This is a cosmological analogue of quantum decoherence. When this happens, quantum fields are well described by classical stochastic functions and we say that the field enters into 'classical evolution'. This is true for light free quantum fields. Canonical massless vector fields, however, do not become classical as their Lagrangian is invariant under conformal transformation to flat space-time. This is a case, for example, with $U(1)$ vector field with minimal kinetic term [62]. In our case, conformal invariance of the vector field is broken by the time varying kinetic function f in Eq. (5). But the question remains whether self-interaction terms do not prevent the non-Abelian field from becoming classical. In Ref. [57] it was shown that after horizon crossing the interacting field does become classical if the interaction is weak; specifically if $\hat{U}$ in Eq. (18) is sufficiently close to unity, $\hat{U} \simeq \hat{1}$. As was discussed before this can only happen if f is a decreasing function in time. With a constraint of the flat perturbation spectrum this means $f \propto a^{-4}$.

In this section we show that correlator functions of the non-Abelian vector field perturbation can be calculated using the classical equation of motion.

B. The Equation of Motion and the Power Spectrum

Extremising the action with the Lagrangian in Eq. (5) we obtain the field equation for non-Abelian vector fields

$$\left[\partial_\lambda + \partial_\lambda \ln \sqrt{-\det[g_{\mu\nu}]}\right]\left(fF_h^{\lambda\kappa}\right) - fg_{\mathrm{c}}\mathrm{f}^{abh}A_\rho^b F_a^{\rho\kappa} = 0, \tag{36}$$

Taking the spatial component of this equation $\kappa = i$ and adopting a temporal gauge $A_0^a = 0$ we get

$$\ddot{A}_i^h + \left(H + \frac{\dot{f}}{f}\right)\dot{A}_i^h - a^{-2}\left(\partial_j\partial_j A_i^h - \partial_i\partial_j A_j^h\right) - a^{-2}g_{\mathrm{c}}\mathrm{f}^{abh}\left[2\left(A_j^a\partial_j A_i^a + \partial_j A_j^a A_i^b\right) - g_{\mathrm{c}}\mathrm{f}^{ade}A_j^b A_j^d A_i^e\right] = 0. \tag{37}$$

We are interested in superhorizon evolution of the vector field perturbation. On these scales derivative terms are negligible and Eq. (37) can be written as

$$\ddot{A}_i^h + \left(H + \frac{\dot{f}}{f}\right)\dot{A}_i^h + g_{\mathrm{c}}^2\mathrm{f}^{abh}\mathrm{f}^{ade}a^{-2}A_j^b A_j^d A_i^e = 0, \tag{38}$$

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[1] For the perturbative approach to be valid $W > \delta W$ must hold. The typical value of the field perturbation is $\delta W \sim \sqrt{\mathcal{P}_+} = H/2\pi$, resulting in $W > H$.

which is the same as that of the homogeneous mode of the field. Changing to the physical canonically normalized vector field in Eq. (8), with $f \propto a^{-4}$, this equation is transformed into

$$\ddot{W}_i^h + 3H\dot{W}_i^h + \frac{g_c^2}{f}\mathfrak{f}^{abh}\mathfrak{f}^{ade}W_j^b W_j^d W_i^e = 0, \tag{39}$$

which is reminiscent of an interacting scalar field. As fields in Eq. (39) are intended to generate the curvature perturbation, they must retain an approximate scale invariance in accordance with observations. For this to be the case W_i^h must be almost a free field, in other words W_i^h must be rolling slowly. In analogy to Ref. [63] we introduce slow-roll parameters

$$\epsilon_{ij}^{ab} \equiv \frac{\dot{W}_i^a \dot{W}_j^b}{2m_{\mathrm{Pl}}H} \quad \text{and} \quad \eta_{ij}^{ab} \equiv \frac{V_{ij}^{ab}}{3H^2}, \tag{40}$$

where $V_{ij}^{ab} \equiv \partial V_i^a/\partial W_j^b$ and

$$V_i^h \equiv \frac{g_c^2}{f}\mathfrak{f}^{abh}\mathfrak{f}^{ade}W_j^b W_j^d W_i^e \propto g^2(t) \tag{41}$$

and require $\left|\epsilon_{ij}^{ab}\right| < 1$ and $\left|\eta_{ij}^{ab}\right| \sim \frac{g_c^2}{3f}\left(\frac{W}{H}\right)^2 < 1$. As structure constants are of order unity, the slow-roll conditions mean that the strength of self-coupling $g(t) \equiv g_c/\sqrt{f(t)}$ in Eq. (4) of canonically normalized field is small, i.e. $g_c^2/f \ll 1$. This is easily achieved when cosmological scales exit the horizon. Because $f \propto a^{-4}$ is an exponentially decaying function with $f(t_s) = 1$, the self-coupling g_c^2/f is exponentially suppressed. Note however that although $\left|\eta_{ij}^{ab}\right| \ll 1$ is easily satisfied when cosmological scales exit the horizon, this condition must hold up until t_s. Even if the evolution becomes strongly non-linear after cosmological scales crosses the horizon, all scales are affected. Thus not only g_c^2/f must be small at horizon crossing but it must remain small when $f(t_s) = 1$, i.e. $g_c^2 < 1$. We assume this to be the case.

Following Ref. [59] we decompose the vector field as

$$W^a = W_0^a + \delta W_1^a + \frac{1}{2}\delta W_2^a + \ldots, \tag{42}$$

where the field notation without space indices means the modulus, e.g. $W^a \equiv |W_i^a|$. The first term in this expression is the homogeneous field. For the rest of the paper we will have no use of the total vector field W_i^a, thus we drop out the subscript '0' from the homogeneous mode and denote it simply by W_i^a. The second term in Eq. (42) is the perturbation and later terms are higher orders in δW_1^a. This expansion is not unique and to determine δW_1^a some auxiliary conditions need to be imposed [59, 64]. We choose δW_1^a in such a way that its equation of motion is linear, i.e. Eq. (39) without the last term. It follows then that δW_1^a obeys the Gaussian statistics and its two point correlation function is

$$\left\langle \delta W_{1i}^a(\mathbf{k}) \delta W_{1j}^b(\mathbf{k}') \right\rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \delta_{ab} T_{ij}^{\mathrm{E}}\left(\hat{\mathbf{k}}\right) \mathcal{P}_+^a(k), \tag{43}$$

where $T_{ij}^{\mathrm{E}}\left(\hat{\mathbf{k}}\right)$ is defined in Eq. (16) and the Fourier transform of δW_{1i}^a is defined in Eq. (9). We also used the fact that there is no correlation between left- and right-handed modes, hence the Kronecker delta δ_{ab}. In de Sitter inflation the power spectrum is scale invariant $\mathcal{P}_+^a = (H/2\pi)^2$. In the slow-roll inflation the spectrum acquires weak scale dependence due to slowly increasing horizon size. Assuming approximately constant $\dot{H}/H^2$ it is

$$\mathcal{P}_+^a = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon}. \tag{44}$$

The spectral tilt '-2ϵ' is due to the slight increase of the Hubble horizon during slow-roll inflation, which is parametrized by $\epsilon \equiv -\dot{H}/H^2$. Because $\epsilon > 0$, each subsequent k mode crosses a horizon of larger size making the amplitude of perturbation smaller.

The spectrum in Eq. (44) is for the non-interacting part δW_1 of the field perturbation. However, the total power spectrum will have additional scale dependence

$$\mathcal{P}_+^a = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon + 2|\eta|}, \tag{45}$$

where $|\eta| = \left|\eta_{ij}^{ab}\right|$ is the modulus of the slow-roll parameter matrix in Eq. (40). The second term in the exponent of the scale dependent factor on the right-hand-side of the above equation is caused by interactions in the last term of Eq. (39). Due to interactions each k mode is not frozen after horizon exit but evolves slowly. As larger modes spend less time outside the horizon they are less affected, which introduces additional k dependence. Both ϵ and $|\eta|$ in Eq. (45) are evaluated at horizon crossing. Although $|\eta| \propto f^{-1}$ is a function of time this does not introduce additional scale dependence as all modes after horizon crossing are affected the same way by the evolution of $|\eta|$. However, $|\eta| \propto f^{-1}$ means that for cosmological scales $|\eta|$ is exponentially suppressed and the ϵ term dominates the spectral tilt. In principle $|\eta|$ introduces an anisotropic scale dependence. But as this term is subdominant, the direction dependence of $|\eta|$ is suppressed.

The presence of non-linear term in Eq. (39) makes the vector field perturbation non-Gaussian. As we have chosen δW_1 to satisfy Gaussian statistics, non-Gaussianity is encapsulated in δW_2. Thus δW_2 satisfies the full non-linear equation. However, the curvature perturbation in the Universe is predominantly Gaussian. So if vector fields are to generate the dominant contribution to the curvature perturbation, they must be predominantly Gaussian too, that is $\delta W_2^a < \delta W_1^a$. With this condition δW_{2i}^a can be seen as the second order perturbation. Thus perturbing Eq. (39) to the second order we find the equation of motion for δW_{2i}^a

$$\delta \ddot{W}_{2i}^h + 3H \delta \dot{W}_{2i}^h + 2V_{2i}^h = 0. \tag{46}$$

Dropping out terms proportional to the slow-roll parameters in Eq. (40) we write

$$V_{2i}^h \equiv \frac{\partial^2 V_i^h}{\partial W_m^a \partial W_n^b} \delta W_{1m}^a \delta W_{1n}^b = \frac{g_c^2}{f} \mathrm{f}^{abh} \mathrm{f}^{adc} \left[W_i^c \delta W_{1j}^b \delta W_{1j}^d + W_j^d \delta W_{1j}^b \delta W_{1i}^c + W_j^b \delta W_{1j}^d \delta W_{1i}^c \right] \tag{47}$$

Assuming slow-roll holds we may also drop the first term in Eq. (46) and write

$$\delta \dot{W}_{2i}^h = -\frac{2V_{2i}^h}{3H}. \tag{48}$$

Taking the Fourier transform it becomes

$$\delta \dot{W}_{2i}^h (\mathbf{k}) = -\frac{2}{3H} \int \frac{\mathrm{d}^3 q_1 \mathrm{d}^3 q_2}{(2\pi)^3} \delta (\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) V_{2i}^h (\mathbf{q}_1, \mathbf{q}_2), \tag{49}$$

where $V_{2i}^h (\mathbf{q}_1, \mathbf{q}_2)$ is

$$V_{2i}^h (\mathbf{q}_1, \mathbf{q}_2, t) = \frac{g_c^2}{f(t)} \mathrm{f}^{abh} \mathrm{f}^{adc} \left[W_i^c \delta W_{1j}^b (\mathbf{q}_1) \delta W_{1j}^d (\mathbf{q}_2) + W_j^d \delta W_{1j}^b (\mathbf{q}_1) \delta W_{1i}^c (\mathbf{q}_2) + W_j^b \delta W_{1j}^d (\mathbf{q}_1) \delta W_{1i}^c (\mathbf{q}_2) \right]. \tag{50}$$

As we are interested in the superhorizon evolution of the field perturbation, to find δW_{2i}^h we integrate Eq. (49) from the horizon exit at t_k, where $k/a(t_k)H = 1$, to some later time t. Because W_i^a is slowly rolling and δW_{1i}^a is constant by definition with $H \approx$ const, the only time dependent term in Eq. (50) is $f \propto a^{-4}$. Thus solving Eq. (49) we find

$$\delta W_{2i}^h (\mathbf{k}, t) = -\frac{1}{6H^2} \int \frac{\mathrm{d}^3 q_1 \mathrm{d}^3 q_2}{(2\pi)^3} \delta (\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) T_{ij}^{\mathrm{E}} (\hat{\mathbf{k}}) V_{2j}^h (\mathbf{q}_1, \mathbf{q}_2, t). \tag{51}$$

From this solution we can find the bound on the strength of self-coupling g_c^2/f for the condition $\delta W_2 < \delta W_1$ to be consistent. Putting $\delta W_1 \sim H$ into Eqs. (50) and (51) it follows $\frac{g_c^2}{f} W < H$. This ensures consistency of using second order perturbation theory to calculate δW_2 and that perturbations of vector fields are predominantly Gaussian.

C. The Three-Point Correlation Function

The three point correlation function from the classical evolution of the field is

$$g_3^{\mathrm{cl}} (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \equiv \left\langle \delta W_l^f (\mathbf{k}_1) \delta W_m^g (\mathbf{k}_2) \delta W_n^h (\mathbf{k}_3) \right\rangle. \tag{52}$$

Because $\delta W_1 \gg \delta W_2$ the largest contribution to g_3^{cl} comes from the term of the form $\left\langle \delta W_1 \delta W_1 \frac{1}{2} \delta W_2 \right\rangle \propto \frac{1}{2} \left\langle \delta W_1 \delta W_1 \delta W_1 \star \delta W_1 \right\rangle$, where a star denotes convolution. As one has to be careful in keeping track of indices we write the dominant term of g_3^{cl} explicitly

$$g_3^{\text{cl}} = \frac{1}{2} \left[\left\langle \delta W_{1l}^f (\mathbf{k}_1) \delta W_{1m}^g (\mathbf{k}_2) \delta W_{2n}^h (\mathbf{k}_3) \right\rangle + \left\langle \delta W_{1l}^f (\mathbf{k}_1) \delta W_{2m}^g (\mathbf{k}_2) \delta W_{1n}^h (\mathbf{k}_3) \right\rangle + \left\langle \delta W_{2l}^f (\mathbf{k}_1) \delta W_{1m}^g (\mathbf{k}_2) \delta W_{1n}^h (\mathbf{k}_3) \right\rangle \right]. \tag{53}$$

To evaluate this expression we use Eq. (51) and Wick's theorem to express four point functions in terms of products of two point ones. After tedious algebra and using Eqs. (43) we obtain

$$g_3^{\text{cl}} = -(2\pi)^3 \delta (\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \mathcal{T}_{lmn}^{(4)fgh} \left(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2, \hat{\mathbf{k}}_3 \right) \frac{\sum_i^3 k_i^3}{\prod_i k_i^3} \frac{g_{\text{c}}^2}{f_{\text{end}}} \frac{4\pi^4 \mathcal{P}_+^2}{12H^2}, \tag{54}$$

where the anisotropy tensor $\mathcal{T}_{lmn}^{(4)fgh}$ is defined in Eq. (25) and we dropped the gauge index from $\mathcal{P}_+^a$ as all vector fields have the same spectrum. Taking $\mathcal{P}_+ = (H/2\pi)^2$ we recover exactly the same result as obtained by a more tedious calculation in the quantum in-in formalism with the dominant term in Eq. (29).

V. THE CURVATURE PERTURBATION

A. δN Formula

In the above we have calculated correlators of the field perturbation. However, field perturbation is not an observable, but the metric perturbation is. We choose a uniform density slicing in which the perturbation of the metric on superhorizon scales is described by the intrinsic curvature ζ. The easiest way to calculate ζ is using the δN formula [65, 66]. This formula was first extended to include vector fields in Ref. [16] and used for non-Abelian fields in Refs. [21, 22]

$$\zeta (\mathbf{x}, t) = N_\phi \delta\phi + N_i^a \delta W_i^a + \frac{1}{2} N_{\phi\phi} (\delta\phi)^2 + N_{\phi i}^a \delta\phi \delta W_i^a + \frac{1}{2} N_{ij}^{ab} \delta W_i^a \delta W_j^b + \dots, \tag{55}$$

where

$$N_\phi \equiv \frac{\partial N}{\partial \phi}, \ N_i^a \equiv \frac{\partial N}{\partial W_i^a}, \ N_{\phi i}^a \equiv \frac{\partial^2 N}{\partial \phi \partial W_i^a} \text{ and } N_{ij}^{ab} \equiv \frac{\partial^2 N}{\partial W_i^a \partial W_j^b}. \tag{56}$$

N in these expressions is the number of e-foldings of local expansion from the initial flat hypersurface to the final uniform density hypersurface at final time t when ζ becomes constant. In Refs. [21, 22] t was taken to be just after the horizon crossing. However, as Eq. (29) shows, the bispectrum of the field perturbation is actually dominated by the interaction of classical fields during classical evolution outside the horizon.

Derivatives in Eq. (56) are taken with respect to homogeneous fields. The precise form of these derivatives depends on the mechanism through which the field perturbation generates ζ. $\delta\phi$ in this equation is the perturbation of some scalar field if any of such fields contribute to the curvature perturbation. In this paper, by keeping only the second and fifth terms of the right-hand-side of Eq. (55) we assume that predominantly vector fields contribute to ζ and any other source is negligible.

B. The Spectrum

Let us first consider the two point correlation function of ζ. In Fourier space we may write

$$\langle \zeta (\mathbf{k}) \zeta (\mathbf{k}') \rangle = N_i^a N_j^b \left\langle \delta W_i^a (\mathbf{k}) \delta W_j^b (\mathbf{k}') \right\rangle \equiv (2\pi)^2 \delta (\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \mathcal{P}_\zeta (\mathbf{k}), \tag{57}$$

where $\delta W_i^a = \delta W_{1i}^a + \frac{1}{2} \delta W_{2i}^a + \dots$. From Eq. (43) we find

$$\mathcal{P}_\zeta (\mathbf{k}) = \sum_{a,b} \delta_{ab} N_i^a N_j^b T_{ij}^{\text{E}} \left(\hat{\mathbf{k}} \right) \mathcal{P}_+^a (k). \tag{58}$$

Due to the presence of $T_{ij}^{\mathrm{E}}\left(\hat{\mathbf{k}}\right)$ in this expression the power spectrum of ζ has an angular modulation. The isotropic part is

$$\mathcal{P}_{\zeta}^{\mathrm{iso}} \equiv \mathcal{P}_+ \sum_a N_a^2, \tag{59}$$

where the sum is over all gauge fields and N_a is the absolute value $N_a \equiv |N_i^a|$. We also used the fact that power spectra $\mathcal{P}_+^a = \mathcal{P}_+$ are the same for all fields. Then the total spectrum of ζ is

$$\mathcal{P}_{\zeta}\left(\mathbf{k}\right) = \mathcal{P}_{\zeta}^{\mathrm{iso}} \left[1 - \frac{\sum_a \left(\mathbf{N}^a \cdot \hat{\mathbf{k}}\right)^2}{\sum_a N_a^2} \right]. \tag{60}$$

If $a = 1$ this expression reduces to Eq. (1) with quadrupole anisotropy of an amplitude $g_{\zeta} = -1$ and such a large anisotropy is ruled out by observations. Thus, a massless $U(1)$ vector field cannot generate the total ζ. However, with the large number of randomly oriented vector fields, the anisotropy is suppressed by the number of fields. To see this, note that N is proportional to vector fields W^a. Since all vector fields satisfy the same equation of motion, assuming similar initial conditions it is reasonable to expect that contributions of all W^a to N are of the same order. In this case N_a's are of the same order too, in particular $N_W \equiv N_a \sim N_b$ for all a and b. Then $\sum_a N_a^2 = \mathcal{N} N_W^2$, where $\mathcal{N}$ is the number of fields and Eq. (60) becomes

$$\mathcal{P}_{\zeta}\left(\mathbf{k}\right) = \mathcal{N} \mathcal{P}_+ N_W^2 \left[1 - \frac{1}{\mathcal{N}} \sum_a^{\mathcal{N}} \left(\hat{\mathbf{W}}^a \cdot \hat{\mathbf{k}}\right)^2 \right], \tag{61}$$

where $\hat{\mathbf{W}}^a$ are unit vectors along the directions of homogeneous vector fields W_i^a and we used the fact that $\hat{\mathbf{W}}^a = \hat{\mathbf{N}}^a \equiv \mathbf{N}^a/N_a$. With $\mathcal{N}$ vector fields, the anisotropic part of the spectrum is a sum of $\mathcal{N}$ quadrupoles. If these are randomly oriented, the anisotropy is suppressed by $\mathcal{N}^{-1}$. Thus, with the large enough non-Abelian symmetry group the total curvature perturbation can be generated solely by vector fields. In view of contradicting conclusions of Refs. [39, 40] we consider two bounds on g_{ζ} to estimate $\mathcal{N}$. If we accept that a systematic effect causing the large anisotropy in the spectrum (0.29 ± 0.031 as claimed in Ref. [39]) is unknown, the bound on the anisotropy in $\mathcal{P}_{\zeta}$ of the primordial origin can be taken to be $g_{\zeta} < 0.29$. In such a case four vector fields $\mathcal{N} = 4$ is enough for this bound to be satisfied. If, on the other hand, the large observed anisotropy in $\mathcal{P}_{\zeta}$ is caused by the WMAP beam asymmetry, as claimed in Ref. [40], then the corrected bound on primordial statistical anisotropy gives $|g_{\zeta}| < 0.07$ [41]. To satisfy this bound $\mathcal{N} \geq 15$ is needed. These estimates for $\mathcal{N}$ are made assuming random orientation of the homogeneous vector fields and their similar magnitudes. If, however, all vector fields are parallel or one of the N_a's is dominant, then $g_{\zeta} = -1$ and such configuration is excluded.

It is also possible to generate statistically isotropic curvature perturbation by considering a triad of orthogonal vector fields with equal norm. This configuration in a context of vector inflation was studied in Refs. [51, 52], where $SU(2)$ group is considered. However, we feel that a scenario with random orientation of larger number of fields is a more natural setup. In addition, such setup also has an advantage of providing observational signature, that is non-negligible statistical anisotropy.

C. The Bispectrum

In this section we calculate the bispectrum B_{ζ} at the end of inflation. In momentum space it is defined by

$$\langle \zeta\left(\mathbf{k}_1\right) \zeta\left(\mathbf{k}_2\right) \zeta\left(\mathbf{k}_3\right) \rangle = \left(2\pi\right)^3 \delta\left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3\right) B_{\zeta}\left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\right). \tag{62}$$

Non-vanishing B_{ζ} is the result of two contributions. The first contribution, let us denote it by $B_{\zeta 1}$, is from the non-Gaussian field perturbations due to self-interactions of the vector fields. The second, $B_{\zeta 2}$, is due to the non-linear terms in the δN formula in Eq. (55).

Let us start by calculating the first contribution. From Eq. (55)

$$\langle \zeta\left(\mathbf{k}_1\right) \zeta\left(\mathbf{k}_2\right) \zeta\left(\mathbf{k}_3\right) \rangle \supset N_l^f N_m^g N_n^h \left\langle \delta W_l^f\left(\mathbf{k}_1\right) \delta W_m^g\left(\mathbf{k}_2\right) \delta W_n^h\left(\mathbf{k}_3\right) \right\rangle. \tag{63}$$

The three point correlation function of the field perturbation was calculated in sections III and IV. Taking the result in Eq. (54) the bispectrum $B_{\zeta 1}$ becomes

$$B_{\zeta 1} = -g_{\text{end}}^2 \frac{4\pi^4}{12H^2} \frac{\sum_i k_i^3}{\prod_i k_i^3} \mathcal{P}_+^2 N_l^f N_m^g N_n^h \mathcal{T}_{lmn}^{(4)fgh} \left(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2, \hat{\mathbf{k}}_3 \right), \tag{64}$$

where $g_{\text{end}} \equiv g_{\text{c}}/\sqrt{f_{\text{end}}}$ is the strength of the self-coupling of canonically normalized vector fields at the end of inflation. Fallowing Refs. [17, 45] we introduce vectors $\mathcal{M}_i^a$ to simplify expressions for the bispectrum

$$\mathcal{M}_j^a (\mathbf{k}) \equiv \mathcal{P}_+ N_i^a T_{ij}^{\text{E}} \left(\hat{\mathbf{k}} \right) = \mathcal{P}_+ N_a \left[\hat{W}_j^a - \hat{k}_j \left(\hat{\mathbf{W}}^a \cdot \hat{\mathbf{k}} \right) \right], \tag{65}$$

where no summation over a is assumed. Using this definition and Eq. (25) the bispectrum in Eq. (64) becomes

$$B_{\zeta 1} = -4\pi^4 \frac{\sum_i k_i^3}{\prod_i k_i^3} \frac{g_{\text{end}}^2}{12H^2} \left(\text{f}^{abh} \text{f}^{agf} + \text{f}^{agh} \text{f}^{abf} \right) W_i^b N_i^g \left[\mathcal{M}_j^f (\mathbf{k}_1) \mathcal{M}_j^h (\mathbf{k}_3) + \text{c.p.} \right]. \tag{66}$$

In this equation 'c.p.' stands for cyclic permutations of vectors $\mathbf{k}$.

The second contribution to the three-point correlator of the curvature perturbation is from non-linear terms in Eq. (55). As we assume that only vector fields generate ζ the dominant term will be

$$\langle \zeta (\mathbf{k}_1) \zeta (\mathbf{k}_2) \zeta (\mathbf{k}_3) \rangle \supset \frac{1}{2} N_{ij}^{ab} N_m^c N_n^d \left\langle \delta W_i^a \star \delta W_j^b (\mathbf{k}_1) \delta W_m^c (\mathbf{k}_2) \delta W_n^d (\mathbf{k}_3) \right\rangle + \text{c.p.} \tag{67}$$

Using Wicks theorem and Eq. (43) the bispectrum from the the above expression becomes

$$B_{\zeta 2} = 4\pi^4 \frac{1}{k_1^3 k_3^3} \mathcal{M}_i^a (\mathbf{k}_1) N_{ij}^{ab} \mathcal{M}_j^b (\mathbf{k}_3) + \text{c.p.} \tag{68}$$

Both $B_{\zeta 1}$ and $B_{\zeta 2}$ depend not only on the absolute values of wavevectors $\mathbf{k}$ but also on their direction, making the bispectrum anisotropic. The total bispectrum is $B_\zeta = B_{\zeta 1} + B_{\zeta 2}$. To evaluate which term is the dominant one note that $N_{ab} \propto N_a/W^b$, where $N_{ab} \equiv \left| N_{ij}^{ab} \right|$. Thus, in order for the first term to dominate, $g_{\text{end}} W > H$ must be satisfied.[2] If this is the case, the slow-roll condition $\left| \eta_{ij}^{ab} \right| < 1$ is violated. Then the evolution of the homogeneous modes of vector fields becomes strongly non-linear and the above calculations do not apply. However, if f is modulated by the inflaton or some degree of freedom which is stabilized at the end of inflation then f_{end} is equal or very close to unity, i.e. $f_{\text{end}} \lesssim 1$ and $g_{\text{end}} \sim g_{\text{c}}$. In this case we can expect $g_{\text{c}} W \sim H$ if W is not much larger than H (see the footnote on page 8), as it is natural for g_{c} to be not much bellow unity in particle physics models. If this is the case, then both contributions to B_ζ are comparable. If, on the other hand, $g_{\text{end}} W < H$ then $B_{\zeta 2}$ contribution to the bispectrum dominates.

VI. THE END-OF-INFLATION SCENARIO

Let us implement the results of previous sections to a specific example using the end-of-inflation scenario. This scenario was suggested in Ref. [67] invoking only scalar fields. In usual hybrid inflation models inflation ends when the waterfall field is destabilized by the inflaton. This happens when the inflaton reaches some critical value and the waterfall field mass becomes tachyonic. As this critical value is determined solely by the inflaton itself, inflation ends on a uniform energy density slice. If, as suggested in Ref. [67], this critical value is modulated by some additional field, then the uniform density slice does no longer coincide with the end-of-inflation slice. This induces the perturbation in the distance between flat and uniform density slice, which is equal to the perturbation in ζ. In Ref. [15] it was shown that if the modulating field is $U(1)$ vector field, the generated ζ is in general statistically anisotropic (see also Ref. [17]). In this example we extend the scenario proposed in Ref. [15] to include non-Abelian vector fields.

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[2] This is in contrast to Refs. [21, 22] where the opposite bound was assumed.

A. The Model

Let us consider a Lagrangian which is invariant under transformations of some non-Abelian symmetry group G

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}fF_a^{\mu\nu}F_{\mu\nu}^a + \frac{1}{2}\text{Tr}\left[\left(D_\mu\Phi\right)^\dagger D^\mu\Phi\right] - V\left(\varphi,\Phi\right), \tag{69}$$

where $\text{Tr}\left[\ldots\right]$ stands for trace, φ is the inflaton field and $F_{\mu\nu}^a$ is the field strength tensor defined in Eq. (6). The gauge kinetic function f may be the function of the inflaton $f\left(\varphi\right)$. This has an advantage that we don't introduce additional degrees of freedom. The behavior of $f\left(\varphi\right)$ for Abelian vector fields was studied in Refs. [32, 33]. It was found that the required scaling $f \propto a^{-4}$ becomes an attractor solution in a large parameter space. Such kinetic function was also studied for non-Abelian vector fields in Ref. [35]. However, for the present purpose we don't need to assume the source of modulation of f.

Φ in Eq. (69) is the Higgs field corresponding to a non-trivial representation of G while the covariant derivative of the Higgs field D_μ is given by

$$D_\mu = \partial_\mu + i\lambda_A \mathbf{T}^a A_\mu^a, \tag{70}$$

where $\mathbf{T}^a$ are generators which satisfy the Lie algebra $\left[\mathbf{T}^a, \mathbf{T}^b\right] = if^{abc}\mathbf{T}^c$ of an unbroken symmetry group G and λ_A is the gauge coupling constant, coupling the Higgs field to the vector gauge fields.

The effective potential V in Eq. (69) is taken to be

$$V\left(\varphi,\Phi\right) = \frac{\lambda}{4}\left[\text{Tr}\left(\Phi^\dagger\Phi\right) - M^2\right]^2 + \frac{\kappa^2}{2}\varphi^2\text{Tr}\left(\Phi^\dagger\Phi\right) + V\left(\varphi\right), \tag{71}$$

where λ, κ and M are constants with M being a symmetry breaking scale. $V\left(\varphi\right)$ is the potential of φ providing the slow-roll inflation. $V\left(\varphi,\Phi\right)$ can be expressed in a more familiar form if we write the Higgs field as

$$\Phi \equiv \phi\mathbf{l}, \tag{72}$$

where $\mathbf{l}$ is the matrix defining the direction of symmetry breaking in the field space with $\text{Tr}\left[\mathbf{l}^\dagger\mathbf{l}\right] = 1$. Then Eq. (71) becomes

$$V\left(\varphi,\phi\right) = \frac{1}{4}\lambda\left(\phi^2 - M^2\right)^2 + \frac{1}{2}\kappa^2\varphi^2\phi^2 + V\left(\varphi\right), \tag{73}$$

which is the potential of the hybrid inflation. But in contrast to the standard hybrid inflation scenarios we assume that the dominant part of the curvature perturbation is not generated by the inflaton field. Instead ζ is generated by the gauge fields through the gauge coupling constants which couple them to the Higgs field in the covariant derivative in Eq. (70). To see this, note from Eqs. (69), (70) and (73) that the effective mass squared of the Higgs field ϕ is

$$m_{\text{eff}}^2\left(\mathbf{x}\right) = \kappa^2\varphi^2 - \lambda M^2 - \lambda_A^2 A_\mu^a A_b^\mu \mathbf{l}^\dagger \mathbf{T}^a \mathbf{T}^b \mathbf{l}. \tag{74}$$

In the unitary gauge $\mathbf{l}$ is such that the last term in this expression is diagonalised to obtain a sum of massive vector fields $\tilde{M}^{ab}\tilde{A}_\mu^a\tilde{A}_b^\mu$. $\tilde{M}^{ab}$ is a diagonal matrix with the only non-zero elements corresponding to broken generators. Note, however, that $\tilde{M}^{ab}$ is not the mass matrix of the vector fields but $\phi^2\tilde{M}^{ab}$ is. Without the loss of generality, we can arrange generators $\mathbf{T}^a$ in such a way that low a's correspond to generators of unbroken subgroup and higher a's correspond to the broken ones. Then $\tilde{M}^{ab}$ will have non-zero elements only in the lower right block, which can be written as $M^{\bar{c}\bar{d}}$, where we used bars over indices to remind us that they run only over the broken generators but not the full group.

B. The Curvature Perturbation

The curvature perturbation in this set up has two contributions

$$\zeta = \zeta_\varphi + \zeta_{\text{e}}. \tag{75}$$

The first contribution ζ_φ is generated at the horizon crossing during the slow-roll inflation. The resulting power spectrum of ζ_φ is well known to be [68]

$$\mathcal{P}_{\zeta_\varphi} = \frac{1}{2m_{\mathrm{Pl}}^2 \epsilon_k} \left(\frac{H_k}{2\pi}\right)^2, \tag{76}$$

where H_k and ϵ_k are the Hubble and slow-roll parameters evaluated at the horizon exit and we used $N_\varphi \equiv \partial N/\partial\varphi = \left(2m_{\mathrm{Pl}}^2 \epsilon_k\right)^{-1/2}$ for the slow-roll inflation. The contribution of ζ_φ to non-Gaussianity is proportional to slow-roll parameters at horizon exit and, therefore, too small to ever be observable [55, 69].

When the inflaton crosses some critical value φ_{c}, at which m_{eff}^2 in Eq. (74) becomes negative, inflation terminates and the Higgs field rolls down to the minimum of the potential. From Eq. (74) we find

$$\varphi_{\mathrm{c}}^2 \equiv \frac{\lambda}{\kappa^2} M^2 - \frac{1}{\kappa^2 f} M^{\bar{a}\bar{b}} W_i^{\bar{a}} W_i^{\bar{b}}, \tag{77}$$

where we made use of the temporal gauge and Eq. (8) to specify φ_{c} in terms of the physical, canonically normalized vector fields $W_i^{\bar{a}}$. The second term in Eq. (77) is subdominant, i.e. $\kappa^2 \varphi_{\mathrm{c}}^2 \approx \lambda M^2$, giving

$$\lambda M^2 \gg M^{\bar{a}\bar{b}} W_i^{\bar{a}} W_i^{\bar{b}}/f. \tag{78}$$

But, due to perturbations of the vector fields, it modulates the critical value of the inflaton, making $\varphi_{\mathrm{c}}(\mathbf{x})$ a function of space coordinates $\mathbf{x}$. Thus the end of inflation hypersurface does not coincide with the uniform energy density hypersurface which results in the generation of the curvature perturbation ζ_{e}. Up to the second order ζ_{e} is given by

$$\zeta_{\mathrm{e}} = N_{\mathrm{e}} \delta\varphi_{\mathrm{c}} + N_{\mathrm{ee}} (\delta\varphi_{\mathrm{c}})^2, \tag{79}$$

where $N_{\mathrm{e}} \equiv \partial N/\partial\varphi_{\mathrm{c}} = \left(2m_{\mathrm{Pl}}^2 \epsilon_{\mathrm{e}}\right)^{-1/2}$, $N_{\mathrm{ee}} \equiv \partial^2 N/\partial\varphi_{\mathrm{c}}^2$ and ϵ_{e} is the first slow-roll parameter at the end of inflation. The perturbation of φ_{c} can be written as [3]

$$\delta\varphi_{\mathrm{c}} = \frac{\partial\varphi_{\mathrm{c}}}{\partial W_i^{\bar{a}}} \delta W_i^{\bar{a}} + \frac{\partial\varphi_{\mathrm{c}}}{\partial f} \delta f, \tag{80}$$

where $\delta f = \dot{f}\, \delta\varphi_{\mathrm{c}}/\dot{\varphi}_{\mathrm{c}}$ is the variation of the kinetic function f corresponding to the time shift from the hypersurface of the uniform energy density to the end of inflation. As we require $f \propto a^{-4}$ for the gauge and self-couplings of the canonically normalized fields to be small, the time derivative of f is negative, $\dot{f} < 0$. Thus, the second term in Eq. (80) suppresses ζ_{e}. Even more so, if this term dominates $\delta\varphi_{\mathrm{c}}$, no perturbation is generated at the end of inflation. To ensure, this does not happen, we require the first term to dominate, which gives the constraint

$$\left(\frac{M^{\bar{a}\bar{b}} W_i^{\bar{a}} W_i^{\bar{b}}}{\kappa^2 f_{\mathrm{e}} \varphi_{\mathrm{c}} m_{\mathrm{Pl}}}\right)^2 \ll \epsilon_{\mathrm{e}}, \tag{81}$$

where f_{e} is evaluated just before the end of inflation. To simplify calculations we assumed a stronger condition, that the second term in Eq. (80) is completely negligible. Then $\delta\varphi_{\mathrm{c}}$ is equal to

$$\delta\varphi_{\mathrm{c}} = -\frac{M^{\bar{a}\bar{b}} W_i^{\bar{b}}}{\kappa^2 f_{\mathrm{e}} \varphi_{\mathrm{c}}} \delta W_i^{\bar{a}}, \tag{82}$$

and the isotropic part of the power spectrum of ζ_{e} in Eq. (59) is given by

$$\mathcal{P}_{\zeta_{\mathrm{e}}}^{\mathrm{iso}} = \mathcal{P}_+ \sum_{\bar{a}} N_{\bar{a}}^2, \tag{83}$$

where from Eqs. (79) and (82)

$$N_i^{\bar{a}} = -N_{\mathrm{e}} \frac{M^{\bar{a}\bar{b}} W_i^{\bar{b}}}{\kappa^2 f_{\mathrm{e}} \varphi_{\mathrm{c}}}. \tag{84}$$

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[3] Note that the second term in this equation was neglected in Refs. [15, 17].

As was mentioned in subsection V B if only one vector field contributes to the curvature perturbation, the anisotropy in the power spectrum is $g_\zeta = -1$ and such a large value is excluded by observations. This is the case, for example, with the Abelian vector field. Thus authors of Refs. [15, 17] assumed that $\zeta_e < \zeta_\varphi$, in which case $\mathcal{P}_\zeta^{\text{iso}}$ is dominated by the scalar field contribution and the subdominant vector field contribution generates anisotropy in the spectrum of ζ with $|g_\zeta| < 1$. In our case, since we are dealing with non-Abelian vector fields, not one but several vector fields contribute to ζ. If their orientation in space is random, the anisotropy in the spectrum is suppressed by the number of fields $\mathcal{N}$ (see Eq. (61)). Thus with the large enough $\mathcal{N}$ (which is evaluated in subsection VI C) we can generate the total curvature perturbation without violating observational bounds on g_ζ. $\mathcal{P}_{\zeta_e}$ dominates the spectrum of the curvature perturbation if $|N_{\bar{a}}| \gg N_\varphi$. Using Eq. (84) this bound becomes

$$\left(\frac{\lambda_A^2}{f_e} \frac{W}{\kappa^2 \varphi_c} \right)^2 \gg \frac{\epsilon_e}{\epsilon_k} = e^{-2N_e \eta}, \tag{85}$$

where λ_A^2/f_e is the gauge coupling of the vector field to the Higgs field at the end of inflation. Evaluating Eq. (85) we assumed that all gauge fields are of the same order, i.e. $W \sim W^{\bar{a}}$ for all $\bar{a}$. We also used the fact that absolute values of matrix elements of generators $\mathbf{T}^a$ in Eq. (70) are of order unity so that $\text{Tr}\left(M^{\bar{a}\bar{b}}\right) \sim \lambda_A^2$. The slow-roll parameter η in Eq. (85) is $\eta \equiv m_{\text{Pl}}^2 V_{\varphi\varphi}(\varphi)/V(\varphi)$ and subscripts denote the second derivative of $V(\varphi)$ with respect φ. As discussed in Ref. [67] with ζ_e dominant, η has nothing to do with the spectral index of the curvature perturbation and can even be $\eta \sim 1$. In this case the right hand side of Eq. (85) can be far bellow unity.

Comparing two bounds in Eq. (81) and (85) we find that for successful end-of-inflation scenario, in which gauge fields generate the dominant contribution to the curvature perturbation, the homogeneous gauge field value must satisfy

$$\left(\frac{m_{\text{Pl}}}{W} \right)^2 \epsilon_e \gg \left(\frac{\lambda_A^2}{\kappa^2} \frac{W}{f_e \varphi_c} \right)^2 \gg \frac{\epsilon_e}{\epsilon_k}. \tag{86}$$

From the first and last terms we find

$$\left(\frac{H}{m_{\text{Pl}}} \right)^2 \ll \left(\frac{W}{m_{\text{Pl}}} \right)^2 \ll \epsilon_k, \tag{87}$$

where the first constraint is explained in the footnote on page 8. Assuming the scale of inflation to be of the order of GUT scale, i.e. $\sim 10^{16}$ GeV, $H/m_{\text{Pl}} \sim 10^{-4}$. Taking ϵ_k, when cosmological scales leave the horizon, to be of order 10^{-2}, the bound in Eq. (87) gives $10^{-4} \ll W/m_{\text{Pl}} \ll 10^{-1}$.

C. Anisotropic Spectrum and Bispectrum

To find the full power spectrum of the curvature perturbation generated by the non-Abelian gauge fields let us substitute Eq. (84) into (83). Using Eq. (60) we find

$$\mathcal{P}_\zeta(\mathbf{k}) = \mathcal{P}_+ C^2 \left(M^2\right)^{\bar{a}\bar{b}} W^{\bar{a}} W^{\bar{b}} \left[1 - \frac{\left(M^2\right)^{\bar{a}\bar{b}} \left(\mathbf{W}^{\bar{a}} \cdot \hat{\mathbf{k}}\right) \left(\mathbf{W}^{\bar{b}} \cdot \hat{\mathbf{k}}\right)}{\left(M^2\right)^{\bar{a}\bar{b}} W^{\bar{a}} W^{\bar{b}}} \right], \tag{88}$$

where $\left(M^2\right)^{\bar{a}\bar{b}} \equiv M^{\bar{a}\bar{c}} M^{\bar{c}\bar{b}}$ is a diagonal matrix, $W^{\bar{a}}$ is the modulus of the vector field $W^{\bar{a}} \equiv |W_i^{\bar{a}}|$ and C is defined as

$$C \equiv \frac{1}{\kappa^2 f_e} \frac{N_e}{\varphi_c}. \tag{89}$$

Note that $\mathcal{P}_{\zeta_e}$ is determined solely by the massive vector fields. If the homogeneous values of all vector fields are of the same order, i.e. $W \sim W^{\bar{a}}$ for all $\bar{a}$, then the power spectrum in Eq. (88) becomes

$$\mathcal{P}_\zeta(\mathbf{k}) \approx \lambda_A^4 \mathcal{N} \mathcal{P}_+ (CW)^2 \left[1 - \frac{1}{\mathcal{N}} \sum_{\bar{a}} \left(\hat{\mathbf{W}}^{\bar{a}} \cdot \hat{\mathbf{k}} \right)^2 \right], \tag{90}$$

where $\mathcal{N}$ is the number of *massive* vector fields and we used $\text{Tr}\left(M^{\bar{a}\bar{b}}\right) \sim \lambda_A^2$. As was discussed after Eq. (61) $\mathcal{N} \geq 15$ or $\mathcal{N} \geq 4$ is needed to avoid observational constraints on g_ζ, depending if the systematics causing detected anisotropy

in the spectrum is believed to be the asymmetry of WMAP beams or unknown. The Lagrangian in Eq. (69) was assumed to be invariant under the transformation of some non-Abelian symmetry group G. To estimate the minimal rank of the group G, which satisfies the bounds on $\mathcal{N}$, let us assume that G is a special unitary group $SU(N)$ which at the phase transition is broken to $SU(N-1)$. Such symmetry breaking results in $\mathcal{N} = 2N - 1$ massive gauge fields. Thus for a weaker bound on $\mathcal{N}$ the $SU(3)$ group already generates $g_\zeta < 0.29$. If, on the other hand, the stronger bound on $\mathcal{N}$ applies (with $|g_\zeta| < 0.07$), at least $SU(8)$ is needed. In realistic particle physics models one has to be careful in choosing a symmetry group as not to overproduce monopoles after the symmetry breaking [70]. However, the above estimate demonstrates that the primordial curvature perturbation can be generated with gauge fields of reasonably large groups. Even more so, the anisotropy in the spectrum g_ζ, generated by such groups, are of the magnitude which will be possible to test in the very near future by the Planck satellite. As it is shown in Refs. [36, 41] the Planck data will allow to constrain g_ζ with an accuracy of 0.01.

To find the bispectrum for the non-Abelian end-of-inflation scenario let us differentiate $N_i^{\bar{a}}$ in Eq. (84) one more time

$$N_{ij}^{\bar{a}\bar{b}} = -\delta_{ij} C M^{\bar{a}\bar{b}} + \frac{N_i^{\bar{a}} N_j^{\bar{b}}}{\varphi_c N_e}. \tag{91}$$

Using Eq. (78) and $\kappa^2 \varphi_c^2 \approx \lambda M^2$ one can easily check that the first term dominates in this expression.

Also let us find the vector $\mathcal{M}_i^a(\mathbf{k})$ introduced in Eq. (65). With $N_i^{\bar{a}}$ calculated in Eq. (84), $\mathcal{M}_i^a(\mathbf{k})$ becomes

$$\mathcal{M}_i^{\bar{a}}(\mathbf{k}) = -\mathcal{P}_+ C M^{\bar{a}\bar{b}} \left[W_i^{\bar{b}} - \hat{k}_i \left(\mathbf{W}^{\bar{b}} \cdot \hat{\mathbf{k}} \right) \right]. \tag{92}$$

The first bispectrum $B_{\zeta 1}$ in Eq. (66), which is due to self interactions of gauge fields, can be calculated using the results in Eqs. (84) and (92). After some algebra we obtain

$$\begin{aligned} B_{\zeta 1} &= 4\pi^4 \frac{\sum_i k_i^3}{\prod_i k_i^3} \frac{g_{\text{end}}^2}{12 H^2} C^3 \mathcal{P}_+^2 \left(\mathrm{f}^{ab\bar{h}} \mathrm{f}^{a\bar{g}\bar{f}} + \mathrm{f}^{a\bar{g}\bar{h}} \mathrm{f}^{ab\bar{f}} \right) M^{\bar{g}\bar{c}} M^{\bar{f}\bar{d}} M^{\bar{h}\bar{e}} \left(\mathbf{W}^b \cdot \mathbf{W}^{\bar{c}} \right) \times \\ &\quad \times \left[\left(\mathbf{W}^{\bar{d}} \cdot \mathbf{W}^{\bar{e}} \right) - 2 \left(\mathbf{W}^{\bar{d}} \cdot \hat{\mathbf{k}}_1 \right) \left(\mathbf{W}^{\bar{e}} \cdot \hat{\mathbf{k}}_1 \right) + \left(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_3 \right) \left(\mathbf{W}^{\bar{d}} \cdot \hat{\mathbf{k}}_1 \right) \left(\mathbf{W}^{\bar{e}} \cdot \hat{\mathbf{k}}_3 \right) + \text{c.p.} \right]. \tag{93} \end{aligned}$$

First, note that the bispectrum is anisotropic due to its dependence on the direction of $\mathbf{k}$ wavevectors. The anisotropy in the bispectrum from self-interactions is solely determined by massive vector fields. The amplitude of $B_{\zeta 1}$ however is determined by vector fields and structure constants of the whole group as it has unbarred indices.

To evaluate the magnitude of $B_{\zeta 1}$ let us assume that all vector fields are of the same order W and $\text{Tr}\left(M^{\bar{a}\bar{b}} \right) \sim \lambda_A^2$. Using the isotropic part of the spectrum in Eq. (90) and assuming that structure constants are of order unity we find

$$B_{\zeta 1}^{\text{iso}} \approx 4\pi^4 \frac{\eta}{2\mathcal{N}} \frac{g_{\text{end}}^2 W^2}{12 H^2} \left(\mathcal{P}_\zeta^{\text{iso}} \right)^2 \left(\frac{f_e \kappa^2 \varphi_c^2}{\lambda_A^2 W^2} \right) \frac{\sum_i k_i^3}{\prod_i k_i^3}. \tag{94}$$

The expression for the second part of the bispectrum $B_{\zeta 2}$ is quite long and the full result is given in Appendix B. But it is easy to see from Eqs. (91), (92) and (68) that it is anisotropic too and it is determined solely by massive vector fields. The isotropic part of $B_{\zeta 2}$ is

$$B_{\zeta 2}^{\text{iso}} \approx -4\pi^4 \frac{\eta}{2\mathcal{N}} \left(\mathcal{P}_\zeta^{\text{iso}} \right)^2 \left(\frac{f_e \kappa^2 \varphi_c^2}{\lambda_A^2 W^2} \right) \frac{\sum_i k_i^3}{\prod_i k_i^3}. \tag{95}$$

Note, that compared to the single field end-of-inflation scenario in Ref. [67], the bispectrum is suppressed by the number of fields. However, as discussed after Eq. (90) we do not expect $\mathcal{N}$ to be too large. Also note, that the bispectrum from the self-interactions $B_{\zeta 1}$ has an additional factor $(g_{\text{end}} W/H)^2$, where g_{end} is the strength of self-coupling of the canonically normalized gauge fields at the end of inflation. Although this factor can not be much larger than one, as it would make the evolution of gauge fields strongly non-linear, it might be not much smaller than unity. Finally as is shown in Eq. (B2) the anisotropy in the bispectrum as in the spectrum is suppressed by the number of massive gauge vector fields.

VII. SUMMARY AND CONCLUSIONS

The possibility of vector fields playing a non-negligible role in the very early Universe is attracting more and more attention both from theorists as well as data analists. The role of vector fields can be to provide either an

anisotropically expanding Universe during inflation or directly affecting or even generating the primordial curvature perturbation ζ, or both. Some works study the possibility of vector fields driving inflation too. Because vector field, in contrast to a scalar field, chooses a preferred direction, the smoking gun of such models is the statistically anisotropic curvature perturbation. The effects of such anisotropy can be observed in temperature and polarization irregularities of the CMB sky. Indeed, with the measurements of Planck satellite, which is currently collecting data, it will be possible to constrain statistical anisotropy at the level of 0.01 [41].

In this paper we studied the curvature perturbation generated by non-Abelian vector fields. Non-Abelian vector fields are one of the main building blocks of the standard model of particle physics and indeed of any gauge theory and their existence is an experimentally confirmed fact. Moreover, theories beyond the standard model contain large numbers of such fields. In this paper we consider massless non-Abelian fields with the Lagrangian of the form $\mathcal{L} = -\frac{1}{4} f(t) F_a^{\mu\nu} F_{\mu\nu}^a$, with $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_c f^{abc} A_\mu^b A_\nu^c$ and f^{abc} being structure constants of a general Lie group. g_c is normalized in such a way that $f(t_s) = 1$ when it is stabilized at some time t_s. The perturbation spectrum for Abelian vector fields is flat if the kinetic function scales as $f \propto a^{-1\pm3}$. In the non-Abelian case however, $f \propto a^2$ corresponds to a strong effective self-coupling $g(t) = g_c/\sqrt{f(t)}$ for the physical vector fields $W_i^a = \sqrt{f} A_i^a/a$, which results in strongly non-linear evolution. For this reason we assume $f \propto a^{-4}$, which is also necessary for the field perturbation to become classical. The requirement for predominantly linear evolution of W_i^a and its perturbation also puts the bound on the self-coupling of fields. As strong non-linearity affects all scales (not only the ones leaving the horizon) the bound $g_c^2 < 1$ must be satisfied.

In Refs. [32, 33] it was shown that the scaling of the form $f \propto a^{-4}$ can be achieved dynamically through the backreaction of vector fields on the evolution of the inflaton, if f is modulated by the inflation and the vector fields are Abelian. However, this induces anisotropic expansion of order the slow-roll parameter ϵ. Anisotropic expansion in its own right introduces an additional source of statistical anisotropy in the curvature perturbation [26]. However, in our analysis, we do not require f to be necessarily modulated by the inflaton. To avoid anisotropic expansion we also assume a negligible contribution of the vector fields to the the overall energy budget during inflation.

With this setup in section III we calculate the bispectrum of the field perturbation resulting from interactions of fields. To calculate the three point correlation function at the tree level we employ the full quantum perturbation formalism, the so called "in-in formalism". The interaction Hamiltonian with the above Lagrangian has two terms, the cubic term with derivative couplings and the quartic term. The contribution from the first one is suppressed by the physical momentum $p \ll H$ as compared to the second term. While the bispectrum from the quartic term is dominated by the classical evolution of fields, i.e. by interactions after a mode exits the horizon.

This being the case, it is much easier to calculate the correlation functions from the homogeneous classical equation of motion. Such calculation is performed in section IV. It is shown that the result from this method is indeed exactly equal to the dominant part from the full calculation using the in-in formalism.

In section V we calculate the spectrum and bispectrum of the curvature perturbation. It is found that the spectrum has angular modulation. However, in contrast to the single vector field case, the anisotropy g_ζ in the spectrum is suppressed by the number of fields (assuming random orientation). Thus, reasonably large groups can generate small but observable g_ζ. In Ref. [41] it is shown that with the Planck data it will be possible to constrain g_ζ with the precision up to 0.01.

The bispectrum of the curvature perturbation has two contributions, one from the non-Gaussian field perturbation, $B_{\zeta 1}$, and the other from non-linearity in generating ζ, $B_{\zeta 2}$. Both of those contributions have an angular modulation and are comparable if $g_{end}W/H \sim 1$, where W is the modulus of the homogeneous part of the vector fields and g_{end} is the self-coupling strength of canonically normalized vector fields at the end of inflation. If this ratio, however, is much larger than 1, the evolution of vector fields is strongly non-linear and above calculations do not apply. In the opposite regime $B_{\zeta 2}$ dominates.

In the last section VI we present an example of a mechanism for vector fields to generate ζ. In this example we consider a scenario in which the curvature perturbation is generated by varying gauge coupling(s), which couple the Higgs field to vector bosons in the covariant derivative. In such models ζ is generated by the vector boson fields corresponding to broken generators after the phase transition. We calculate the spectrum and the bispectrum. The anisotropy in both of them is suppressed by the number of massive vector gauge bosons.

Appendix A: Calculation of Integrals $I^{(4)}$ and $I^{(3)}$

In this appendix we show how to calculate integrals in Eqs. (26) and (33). Let us rewrite here the first integral

$$I^{(4)} = \frac{g_c^2}{f_0} \text{Re} \left[i \int_{\tau_0}^{\tau_{\text{end}}} d\tau'\, a^8 \left(1 - ik_1\tau'\right)\left(1 - ik_2\tau'\right)\left(1 - ik_3\tau'\right) e^{ik_t\tau'} \right]. \tag{A1}$$

During quasi-de Sitter inflation with $H \approx$ constant the scale factor is $a \approx -1/\tau H$. Using this and denoting $x \equiv k_t\tau$, the above integral can be rewritten as

$$I^{(4)} = \frac{g_c^2}{f_0} k_t^7 H^8 \int_{x_0}^{x_{\text{end}}} \left(-\frac{\sin x}{x^8} + \frac{\cos x}{x^7} + K_1 \frac{\sin x}{x^6} - K_2 \frac{\cos x}{x^5} \right) dx, \tag{A2}$$

where $x_0 \equiv k_t\tau_0 \to -\infty$ corresponds to the initial time when modes are deep within the horizon and $x_{\text{end}} \equiv k_t\tau_{\text{end}} \to 0$ is at the end of inflation. Note that assuming all three k's leave the horizon at similar time, $N_k = -\ln(|x_{\text{end}}|)$ is the number of e-folds from the horizon crossing to the end of inflation. K_1 and K_2 are defined in Eq. (28).

The total integral in Eq. (A2) is the superposition of integrals $\int \sin x/x^n\, dx$ and $\int \cos x/x^n\, dx$, with n being a natural number. The order of n within each integral can be reduced integrating by parts until we arrive at superposition of terms $\sin x_{\text{end}}/x_{\text{end}}^n$ and $\cos x_{\text{end}}/x_{\text{end}}^n$ with appropriate constants and the integral $\int \cos x/x\, dx$. The last one can be evaluated as follows. Let us write

$$\int_{x_0}^{x_{\text{end}}} \frac{\cos x}{x} dx = \int_{-x_{\text{end}}}^{1} \frac{1 - \cos x}{x} dx - \int_{1}^{-x_0} \frac{\cos x}{x} dx - \int_{-x_{\text{end}}}^{1} d\ln x. \tag{A3}$$

Taking the limit $x_{\text{end}} \to -\infty$ and $x_0 \to 0$ first two terms in Eq. (A3) are equal to the Euler-Mascheroni's constant $\gamma \approx 0.577$ and the last term is $-N_k$. Expanding the result around $x_{\text{end}} \to 0$ and neglecting terms proportional to x_{end}^n with $n > 0$, we arrive at the final expression in Eq. (27).

The same method can be used to evaluate $I^{(3)}$. The difference is, that integrating by parts the lowest order integral becomes

$$\int_{x_0}^{x_{\text{end}}} \frac{\sin x}{x} dx = \frac{\pi}{2}, \tag{A4}$$

where we have taken a limit $x_0 \to -\infty$ and $x_{\text{end}} \to 0$.

Appendix B: The Bispectrum $B_{\zeta 2}$

The expression for the second part of the bispectrum $B_{\zeta 2}$ can be calculated using the first, dominant term of $N_{ij}^{\bar{a}\bar{b}}$ in Eq. (91)

$$\mathcal{M}_i^a (\mathbf{k}_1) N_{ij}^{ab} \mathcal{M}_j^b (\mathbf{k}_3) = -\mathcal{P}_+^2 C^3 \left(M^3\right)^{\bar{c}\bar{d}} \Big[\left(\mathbf{W}^{\bar{c}} \cdot \mathbf{W}^{\bar{d}}\right) - \left(\mathbf{W}^{\bar{c}} \cdot \hat{\mathbf{k}}_1\right)\left(\mathbf{W}^{\bar{d}} \cdot \hat{\mathbf{k}}_1\right) -$$
$$- \left(\mathbf{W}^{\bar{c}} \cdot \hat{\mathbf{k}}_3\right)\left(\mathbf{W}^{\bar{d}} \cdot \hat{\mathbf{k}}_3\right) + \left(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_3\right)\left(W^{\bar{c}} \cdot \hat{\mathbf{k}}_1\right)\left(W^{\bar{d}} \cdot \hat{\mathbf{k}}_3\right) \Big], \tag{B1}$$

where $\left(M^3\right)^{\bar{a}\bar{d}} \equiv M^{\bar{a}\bar{b}} M^{\bar{b}\bar{c}} M^{\bar{c}\bar{d}}$ is the diagonal matrix. Inserting this result into Eq. (68) gives

$$B_{\zeta 2} = -4\pi^4 \frac{\left(\mathcal{P}_\zeta^{\text{iso}}\right)^2}{\mathcal{N}\varphi_c N_e} \left(\frac{\lambda M^2}{\lambda_A^2 W^2/f_e}\right) \frac{\sum_i k_i^3}{\prod_i k_i^3} \left\{ 1 - \frac{1}{\mathcal{N}} \sum_{\bar{a}} \left[k_2^3 \left(\hat{\mathbf{W}}^{\bar{a}} \cdot \hat{\mathbf{k}}_1\right)^2 - k_2^3 \left(\hat{\mathbf{W}}^{\bar{a}} \cdot \hat{\mathbf{k}}_3\right)^2 - \right.\right.$$
$$\left.\left. - k_2^3 \left(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_3\right)\left(\hat{\mathbf{W}}^{\bar{a}} \cdot \hat{\mathbf{k}}_1\right)\left(\hat{\mathbf{W}}^{\bar{a}} \cdot \hat{\mathbf{k}}_3\right) + \text{c.p.} \right] / \sum_i k_i^3 \right\}, \tag{B2}$$

where we assumed $W \sim W^{\bar{a}}$ for all $\bar{a}$ and $\text{Tr}\left[\left(M^3\right)^{\bar{a}\bar{b}}\right] \sim \lambda_A^6$. Note that the anisotropy in the bispectrum is suppressed by the number of fields, the same suppression as in the spectrum.

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